Problem 3.28

Let $\hat{D} = d/dx$ (the derivative operator). Find

(a) $\left(\sin \hat{D}\right) x^5$.

(b) $\left(\dfrac{1}{1-\hat{D}/2}\right) \cos(x)$.

Solution

Within each set of parentheses is a function that can be expanded as a power series.

$$\begin{aligned}
\left(\sin \hat{D}\right) x^5 &= \left(\hat{D} - \frac{\hat{D}^3}{6} + \frac{\hat{D}^5}{120} - \frac{\hat{D}^7}{5040} + \cdots\right) x^5 \\
&= \left(\frac{d}{dx} - \frac{1}{6}\frac{d^3}{dx^3} + \frac{1}{120}\frac{d^5}{dx^5} - \frac{1}{5040}\frac{d^7}{dx^7} + \cdots\right) x^5 \\
&= \frac{d}{dx}(x^5) - \frac{1}{6}\frac{d^3}{dx^3}(x^5) + \frac{1}{120}\frac{d^5}{dx^5}(x^5) - \frac{1}{5040}\frac{d^7}{dx^7}(x^5) + \cdots \\
&= (5x^4) - \frac{1}{6}(60x^2) + \frac{1}{120}(120) - \frac{1}{5040}(0) + \cdots \\
&= 5x^4 - 10x^2 + 1
\end{aligned}$$

$$\begin{aligned}
\left(\frac{1}{1-\frac{\hat{D}}{2}}\right)\cos x &= \left[1 + \left(\frac{\hat{D}}{2}\right) + \left(\frac{\hat{D}}{2}\right)^2 + \left(\frac{\hat{D}}{2}\right)^3 + \left(\frac{\hat{D}}{2}\right)^4 + \left(\frac{\hat{D}}{2}\right)^5 + \cdots\right]\cos x \\
&= \left(1 + \frac{\hat{D}}{2^1} + \frac{\hat{D}^2}{2^2} + \frac{\hat{D}^3}{2^3} + \frac{\hat{D}^4}{2^4} + \frac{\hat{D}^5}{2^5} + \cdots\right)\cos x \\
&= \left(1 + \frac{1}{2^1}\frac{d}{dx} + \frac{1}{2^2}\frac{d^2}{dx^2} + \frac{1}{2^3}\frac{d^3}{dx^3} + \frac{1}{2^4}\frac{d^4}{dx^4} + \frac{1}{2^5}\frac{d^5}{dx^5} + \cdots\right)\cos x \\
&= 1(\cos x) + \frac{1}{2^1}\frac{d}{dx}(\cos x) + \frac{1}{2^2}\frac{d^2}{dx^2}(\cos x) + \frac{1}{2^3}\frac{d^3}{dx^3}(\cos x) + \frac{1}{2^4}\frac{d^4}{dx^4}(\cos x) \\
&\quad + \frac{1}{2^5}\frac{d^5}{dx^5}(\cos x) + \frac{1}{2^6}\frac{d^6}{dx^6}(\cos x) + \frac{1}{2^7}\frac{d^7}{dx^7}(\cos x) + \frac{1}{2^8}\frac{d^8}{dx^8}(\cos x) + \cdots \\
&= \cos x + \frac{1}{2^1}(-\sin x) + \frac{1}{2^2}(-\cos x) + \frac{1}{2^3}(\sin x) + \frac{1}{2^4}(\cos x) + \cdots \\
&\quad + \frac{1}{2^5}(-\sin x) + \frac{1}{2^6}(-\cos x) + \frac{1}{2^7}(\sin x) + \frac{1}{2^8}(\cos x) + \cdots \\
&= \cos x\left(1 + \frac{1}{2^4} + \frac{1}{2^8} + \cdots\right) + (-\sin x)\left(\frac{1}{2^1} + \frac{1}{2^5} + \frac{1}{2^9} + \cdots\right) \\
&\quad + (-\cos x)\left(\frac{1}{2^2} + \frac{1}{2^6} + \frac{1}{2^{10}} + \cdots\right) + \sin x\left(\frac{1}{2^3} + \frac{1}{2^7} + \frac{1}{2^{11}} + \cdots\right)
\end{aligned}$$

Factor sine and cosine and then evaluate the geometric series.

$$\begin{aligned}
\left(\frac{1}{1-\frac{\hat{D}}{2}}\right)\cos x &= \cos x\left(1-\frac{1}{2^2}+\frac{1}{2^4}-\frac{1}{2^6}+\frac{1}{2^8}-\frac{1}{2^{10}}+\cdots\right)-\frac{\sin x}{2}\left(1-\frac{1}{2^2}+\frac{1}{2^4}-\frac{1}{2^6}+\frac{1}{2^8}-\frac{1}{2^{10}}+\cdots\right) \\
&= \cos x\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{2n}}-\frac{\sin x}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{2n}} \\
&= \cos x\sum_{n=0}^{\infty}\left(-\frac{1}{4}\right)^n-\frac{\sin x}{2}\sum_{n=0}^{\infty}\left(-\frac{1}{4}\right)^n \\
&= \cos x\left[\frac{1}{1-\left(-\frac{1}{4}\right)}\right]-\frac{\sin x}{2}\left[\frac{1}{1-\left(-\frac{1}{4}\right)}\right] \\
&= \cos x\left(\frac{4}{5}\right)-\frac{\sin x}{2}\left(\frac{4}{5}\right) \\
&= \frac{4}{5}\cos x-\frac{2}{5}\sin x
\end{aligned}$$